OUTFLOW OF A GAS JET FROM A VESSEL OF FINITE SIZE

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Fal'kovich [1] has derived an exact solution of the problem of the outflow of a compressed gas with subsonic velocity w_1 in a jet from a symmetrical rectangular vessel of finite width H through an orifice of size h (Fig. 1). This note presents a detailed analysis of the flow field described by this solution. For $\tau_0 < \tau \leqslant \tau_1$ the latter has the form:

$$\psi(\tau, \theta) = -\frac{Q}{\pi}\theta - \frac{Q}{\pi}\sum_{n=1}^{\infty}\frac{X_n(\tau)}{n}\sin 2n\theta \tag{1}$$

$$X_n(\tau) = \frac{Z_n(\tau)}{Z_n(\tau_1)} - \frac{\tau_0}{(1-\tau_0)^\beta}\,\frac{\zeta_n(\tau_2)Z_n(\tau) - \zeta_n(\tau)Z_n(\tau_1)}{nZ_n(\tau_1)}\,Z_n'(\tau_0)$$

$$\tau = \frac{w^2}{w_{\max}^2}, \qquad \beta = \frac{1}{\varkappa - 1}, \qquad \tau_0 = \frac{w_0^2}{w_{\max}^2}, \qquad \tau_1 = \frac{w_1^2}{w_{\max}}$$

where ψ is the stream function, θ is the angle formed by the velocity

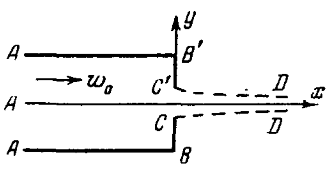

Fig. 1.

vector w with the x-axis, κ is the adiabatic exponent, and τ_0 and τ_1 are the velocities of the gas corresponding to the interior of the vessel far from the orifice and in the jet respectively.

If by h' we designate the width of the jet at infinity, for the outflow Q we have

$$Q = w_1(1-\tau_2)^\beta h' \tag{2}$$

The quantities $Z_n(\tau)$ and $\zeta_n(\tau)$ introduced in (1) are linearly independent integrals of the equation

$$\tau^2(1-\tau)y_n'' + \tau[1+(\beta-1)\tau]y_n' - n^2[1-(2\beta+1)\tau]y_n = 0 \tag{3}$$

the first of which is known in gas dynamics as the Chaplygin function [2], bounded at the point $\tau = 0$.

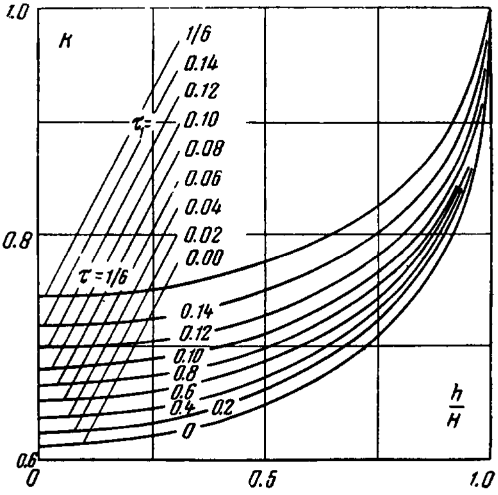

Fig. 2.

1. Following from solution (1), the coefficient of contraction of the jet is given in form [1]

$$k = \frac{h'}{h} = \frac{H}{h}\sqrt{\frac{\tau_0}{\tau_1}}\left(\frac{1-\tau_0}{1-\tau_1}\right)^\beta$$

$$\frac{h}{H} = \frac{1}{k_\infty}\sqrt{\frac{\tau_0}{\tau_1}}\left(\frac{1-\tau_0}{1-\tau_1}\right)^\beta + \frac{8}{\pi}\sqrt{\frac{\tau^3_0}{\tau_1}}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\frac{Z_n'(\tau_0)}{Z_n(\tau_1)} \tag{4}$$

and k_∞ indicates the coefficient of contraction for infinite width of the vessel, $\tau_0 = 0$ $(H = \infty)$, analyzed in detail by Lighthill [3]. Thus (4) shows the influence of the width of the vessel H on the form of the jet in all ranges of speed of the gas within it $(0 \leqslant \tau_1 \leqslant \tau_s)$, including the speed of sound τ_s $(2\beta + 1)^{-1}$ (for air, $\tau_s = 1/6$). In particular, for incompressible liquids

$$Z_n(\tau) = \tau^n, \qquad \beta = 0$$

and as a result of the summing of the series of (4), it is not hard to obtain

$$\frac{h}{H} = \frac{1}{k_\infty}\sqrt{\frac{\tau_0}{\tau_1}} + \frac{2}{\pi}\left(\operatorname{arc\,tg}\sqrt{\frac{\tau_0}{\tau_1}} - \sqrt{\frac{\tau_0}{\tau_1}} - \frac{\tau_0}{\tau_1}\operatorname{arc\,tg}\sqrt{\frac{\tau_0}{\tau_1}}\right), \quad k = \frac{H}{h}\sqrt{\frac{\tau_0}{\tau_1}} \tag{5}$$

The computations according to formula (5) and (4) were carried out with precision up to the fourth place after the decimal point. The results of the computation are presented in Tables 1 and 2; and in Fig. 2 they are presented graphically.

In the computations, the values of the functions $Z_n(\tau)$ and $Z_n'(\tau)$ were taken from the tables of Lighthill [3], in which $n \leqslant 7$. For $n > 7$ the asymptotic formulas of Lighthill [4]

$$Z_n(\tau) = \left[\frac{(1-\tau)^{2\beta+1}}{1-\tau/\tau_s}\right]^{1/4} e^{2nS}\left\{1+O\left(\frac{1}{n}\right)\right\} \quad \text{при } \tau<\tau_s$$

$$Z_n(\tau_s) = \alpha(2n)^{1/6} e^{2n\sigma}\left\{1+O\left(\frac{1}{n}\right)\right\} \tag{6}$$

where

$$S = \sigma + \frac{1}{\sqrt{\tau_s}} \operatorname{arth}\left(\frac{\tau_s-\tau}{1-\tau}\right)^{1/2} - \operatorname{arth}\left(\frac{1-\tau/\tau_s}{1-\tau}\right)^{1/2}$$

$$\sigma = -\frac{1}{\sqrt{\tau_s}} \operatorname{arth}\sqrt{\tau_s} - \frac{1}{2}\ln\beta$$

$$\alpha = 2^{\frac{2\varkappa-1}{2(\varkappa-1)}}(\varkappa+1)^{-\frac{\varkappa+2}{6(\varkappa-1)}}\sqrt{\pi}\, 3^{-1/3}\left[\Gamma\left(\frac{2}{3}\right)\right]^{-1}$$

and of Seifert [5] were used.

$$Z_n'(\tau) = \frac{2n}{\tau}(1-\tau)^{\beta/2-1/4}\left(1-\frac{\tau}{\tau_s}\right)^{1/4} e^{2nS}\left\{1+O\left(\frac{1}{n}\right)\right\} \quad \text{при } \tau<\tau_s \tag{7}$$

In view of the inadequacy of the tables (3) of $Z_n'(0.0025)$ and $Z_n'(0.01)$ for $n \leqslant 7$, they have been computed directly from the hypergeometric series which comprise these functions (2). In the case of τ_0 nearly equal to τ_1, the convergence of the series in (4) was improved by the method of deriving an asymptotic series from it. This latter, as can readily be verified by means of formulas (6) and (7), up to a multiplicative constant has the form

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-2n(S_0-S_1)} = \ln\left[1+e^{-2(S_0-S_1)}\right]$$

2. As has been indicated, for sonic flow $\tau_1 = \tau_s$ the jet straightens out at finite distance from the orifice, and downstream of the orifice a constant flow is established with a velocity equal to that of sound, $W_1 = a_*$. This distance may be determined by use of solution (1) and the formula for transformation to the physical plane (2)

$$dx = \frac{\cos\theta}{w}\,d\varphi - (1-\tau)^{-\beta}\frac{\sin\theta}{w}\,d\psi \tag{8}$$

TABLE 1

τ_1	$\frac{\tau_0}{\tau_2}$	h/H	k
0 Incompressible liquid	0	0	0.6110
	0.05	0.3577	0.6249
	0.20	0.6614	0.6761
	0.40	0.8479	0.7460
	0.60	0.9425	0.8219
	0.30	0.9885	0.9048
	1.00	1.0000	1.0000

If we take into account Chaplygin's equation [2]

$$\frac{\partial \varphi}{\partial \theta} = \frac{2\tau}{(1-\tau)^{\beta}} \frac{\partial \psi}{\partial \tau}$$

in the jet, equation (8) gives

$$dx = \frac{\cos\theta}{a_*} \frac{2\tau_s}{(1-\tau_s)^{\beta}} \left.\frac{\partial \psi}{\partial \tau}\right|_{\tau=\tau_s} d\theta$$

Introducing (1) at this point, and integrating over the segment $(0, \pi/2)$, after computation we obtain the following formula for the length of the portion of the jet in which equalization takes place:

$$x_* = \frac{4\tau_s}{(1-\tau_s)^{\beta}} \frac{Q}{\pi a_*} \sum_{n=1}^{\infty} \frac{X_n'(\tau_s)}{4_n{}^2 - 1}$$

or, as a result of the substitution of (2),

$$\frac{x_*}{h} = \frac{4k\tau_s}{\pi} \sum_{n=1}^{\infty} \frac{X_n'(\tau_s)}{4n^2 - 1} \qquad (9)$$

Whereupon, using the fact that the Wronskian of two linearly independent solutions of (3), Z_n and ζ_n is expressed in the form

$$W(Z_n, \zeta_n) = \frac{(1-\tau)^{\beta}}{\tau} n$$

the function $X_n(\tau_s)$ may be introduced

$$X_n'(\tau_s) = \frac{Z_n'(\tau_s)}{Z_n(\tau_s)} - \frac{\tau_0}{\tau_1}\left(\frac{1-\tau_s}{1-\tau_0}\right)^{\beta} \frac{Z_n'(\tau_0)}{Z_n(\tau_s)}$$

Hence, in the particular case of infinite width of the vessel ($\tau_0 = 0$)

$$\frac{x_*}{h} = \frac{4k_\infty \tau_s}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \frac{Z_n'(\tau_s)}{Z_n(\tau_s)} \qquad (10)$$

TABLE 2.

τ_1	τ_0	h/H	k	τ_1	τ_0	h/H	k
	0	0	0.6233		0	0	0.6364
	0.0025	0.5543	0.6668		0.0025	0.4184	0.6580
0.02	0.01	0.9091	0.7973	0.04	0.01	0.7312	0.7246
	0.02	1.0000	1.0000		0.02	0.9174	0.8115
					0.04	1.0000	1.0000
	0	0	0.6504		0	0	0.6654
	0.0025	0.3559	0.6653		0.0025	0.3196	0.6770
	0.01	0.6548	0.7097		0.01	0.5968	0.7116
0.06	0.02	0.8340	0.7683	0.08	0.02	0.7740	0.7565
	0.04	0.9725	0.8849		0.04	0.8432	0.7818
	0.06	1.0000	1.0000		0.06	0.9878	0.9251
					0.08	1.0000	1.0000
	0	0	0.6815		0	0	0.6988
	0.0025	0.2957	0.6914		0,0025	0.2792	0.7073
	0.01	0.5574	0.7200		0.01	0.5291	0.7324
0.10	0.02	0.7308	0.7572	0.12	0.02	0.6988	0.7646
	0.04	0.8979	0.8277		0.04	0.8698	0.8251
	0.06	0.9672	0.8928		0.06	0.9478	0.8798
	0.10	1.0000	1.0000		0.08	0.9831	0.9281
					0.12	1.0000	1.0000
	0	0	0.7175		0	0	0.7447
	0.0025	0.2670	0.7252		0.0025	0.2554	0.7516
0.14	0.1	0.5108	0.7438	$\frac{1}{6}$	0.01	0.4882	0.7719
	0.02	0.6747	0.7766		0.02	0.6513	0.7977
	0.04	0.8476	0.8303		0.04	0.8254	0.8454
	0.08	0.9721	0.9204		0.06	0.9135	0.8877
	0.10	0.9908	0.9557		0.08	0.9601	0.9241
	0.14	1.0000	1.0000		1/6	1.0000	1.0000

TABLE 3.

τ_0	h/H	x_*/h
0	0	0.6007
0.0025	0.2554	0.6007
0.01	0.4882	0.6007
0.02	0.6513	0.6006
0.04	0.8254	0.5911
0.06	0.9135	0.5677
0.08	0.9601	0.5307
1/6	1.00000	0

The computations according to formulas (9) and (10) are carried out in a manner analogous to Section 1. The value of the function $Z_n'(r_s)/Z_n(r_s)$ is derived from the author's asymptotic expression [6]

$$\frac{Z_n'(\tau_s)}{Z_n(\tau_s)} = -\left\{\frac{v'(0)}{v(0)}(2n)^{2/3} + \frac{2\kappa+5}{10(\kappa+1)^{1/3}} + O(n^{-4/3})\right\}\frac{(\kappa+1)^{1/3}}{2\tau_s}$$

where $v(x)$ is Airy's function [7]. For improving the convergence of series (10), the method of evaluating an asymptotic series was used, which, as follows from the previous case, has the form, up to a multiplicative constant,

$$\frac{v'(0)}{v(0)}\frac{(\kappa+1)^{1/3}}{2^{1/3}}\sum_{n=1}^{\infty}\frac{1}{n^{4/3}} + \frac{2\kappa+5}{20}\sum_{n=1}^{\infty}\frac{1}{n^2} =$$

$$= \frac{v'(0)}{v(0)}\left(\frac{\kappa+1}{2}\right)\zeta\left(\frac{4}{5}\right) + \frac{2\kappa+5}{20}\zeta(2)$$

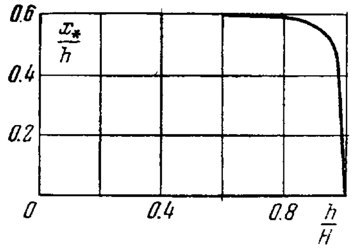

Fig. 3.

where $\zeta(x)$ is the well-known *zeta* function of Riemann [8]. The results of the computation are presented in Table 3 and in Fig. 3. They show interesting properties in the magnitude of the outflow of the constant sonic flow from the orifice. In particular, with decrease of width of the vessel to $h/H = 0.65$, the ratio x_*/h remains constant, and only with the approach of the vessel towards a tube ($h = H$) does it fall rapidly to 0.

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